Lectures on Modality, Day 1

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1 Introduction

§1 Kratzer (1991): "Modality has to do with necessity and possibility."

§2 Hughes and Cresswell (1996): "Modal logic is the logic of necessity and possibility, of 'must be' and 'may be'."

§3 Palmer (2001): "In notional terms [tense, aspect and modality] are, in some way, concerned with the event or situation that is being reported by the utterance ... Tense, rather obviously, is concerned wih the time of the event, while aspect is concerned with the nature of the event, particularly in terms of its 'internal temporal constituency' ... *Modality is concerned with the status of the proposition that describes the event.*" (emphasis added)

§4 Modality is about relations of *consequence* and *consistency*.

§5 Evaluation of a sentence p relative to a *body of information* A.

§6 Variation along several dimensions:

- Nature of the information in A;
- Direction and strength of the relationship between A and p;
- (Speaker's) confidence in the relationship
- etc.

1.1 Some typical modal expressions (in English):

(1) Auxiliaries: *must, may, might, can, could, shall, should, will, would, have to, ought to, ...*

(2) Adjectives: *invincible, permissible, payable, fragile, mortal, edible, fertile, ...*

(3) Adverbs: *presumably, possibly, probably, necessarily, ...*

(4) "Inherent modality": *This room seats fifty people; this car goes eighty miles per hour; ...*

(5) Certain grammatical categories.
  a. Progressive: *John was drawing a circle* (Dowty, 1977)
  b. Temporal conjunctions: *Mary defused the bomb before it exploded* (Beaver and Condoravdi, 2003)
  c. Present tense: *John submits his paper* (Kaufmann, 2005)
  d. etc.

The semantic field of modality is closely related to *propositional attitudes* and *evidentiality*.

- Propositional attitudes:

  (6) '*believe/doubt/wish/suppose (that) p*'

  In many cases, modal expressions depend for their truth or falsehood on facts about propositional attitudes. But the two categories should be kept distinct. Unlike (6), modal expressions are not directly statements *about* propositional attitudes.

- Evidentiality: Markers of the *source* of the information in A. In English, (7a) is a clear case, but (7b) is somewhat under debate.

  (7) a. That restaurant is supposed to be good.
      b. You must be tired!

  The relationship between evidentiality and modality is not clear. Some (notably Palmer, 2001) treat evidentials as modals. For recent writings in this area, sse Garrett (2001) and Faller (2004).

Table 1: Some modal bases

Term	Content	Meaning of "necessity"
Circumstantial	(relevant facts)	*'p is the case'*
Epistemic	(knowledge)	*'p is known'*
Doxastic	(beliefs)	*'p is believed'*
Stereotypical	(normal course of events)	*'p is normal'*
Deontic	(obligations, laws)	*'is required to p'*
Volitive	(decisions)	*'is willing to p'*
Dispositional	(abilities)	*'cannot but p'*
Buletic/Desiderative	(desires, preferences)	*'wants p'*
Teleological	(strategies, goals)	*'aims at p'*
...		

1.2 Major dimensions of variation

Modal base. (Body of information A) This can nicely be illustrated with the various senses of "necessity." The statement that *'p is necessary'* (written '$\Box p$') takes on different meanings for different modalities. Table 1 is a collection of examples from the literature.[1]

Digression on "circumstantial" modality. As far as I know, "circumstantial" modality was introduced by Kratzer (1981). The difference between epistemic and circumstantial modal bases can be subtle. Here is an example from Kratzer (1991):

(8) a. Hydrangeas can grow here.
  b. There might be hydrangeas growing here.

> The two sentences differ in meaning in a way which is illustrated by the following scenario.
> *Hydrangeas*
> Suppose I acquire a piece of land in a far away country and discover that soil and climate are very much like at home, where hydrangeas prosper everywhere. Since hydrangeas are my favorite plants, I wonder whether they would grow in this place and inquire about it. The answer is (8a). In such a situation, the proposition expressed by (8a) is true. It is true regardless of whether it is or isn't likely that there are already hydrangeas in the country we are considering. All that matters is climate, soil, the special properties of hydrangeas, and the like.

[1] Take this with a grain of salt—the terminology is extremely messy in this area.

> Suppose now that the country we are in has never had any contacts whatsoever with Asia or America, and the vegetation is altogether different from ours. Given this evidence, my utterance of (8b) would express a false proposition. What counts here is the complete evidence available. And this evidence is not compatible with the existence of hydrangeas.

(8a): circumstantial

(8b): epistemic

Kratzer (1991) further writes:

> Epistemic modality is the modality of curious people like historians, detectives, and futurologists. Circumstantial modality is the modality of rational agents like gardeners, architects and engineers. A historian asks what might have been the case, given all the available facts. An engineer asks what can be done given certain relevant facts.

End of digression.

Modal force. (Strength of the connection between A and p): *Necessity*, *Possibility*, *Likelihood*, ...

Clearly, modal force is a matter of degree. However, standard modal logic only deals with necessity and possibility—there are no degrees.

There are some extensions, however, among them the "ordering sources" of Kratzer (1981), which gives us a notion of *relative* modality: It lets us express that p is more (less) likely than q. We will deal with them in detail.

Subtleties. There are many differences and idiosyncracies which can only be characterized as lexical properties of particular modals. Most of them are still not well-understood, even in English, let alone other languages. Here are some examples:

- With future reference and a non-stative embedded predicate, '*may*' can be epistemic or deontic (9a), whereas '*must*' is virtually always deontic (9b).

  (9) a. John may leave tomorrow.
      b. John must leave tomorrow.

On the other hand, with past reference, both *'may'* and *'must'* tend to be epistemic:

(10) a. John may have left yesterday.
b. John must have left yesterday.

(It is hard, though not impossible, to get a deontic reading for (10).)

There is an obvious semantic explanation for this: Deontic modals must be *future-oriented* in a certain sense, since you cannot tell someone to have done something. But note that it *is* ok to issue permissions etc. relative to past behavior, so long as the permission is relevant for future behavior:

(11) The successful candidate must have spent at least one year abroad.

- *'May'* and *'can'* are interchangeable on the deontic reading only (i.e., the form with *'can'* disambiguates):

(12) a. John may leave when you do that.
b. John can leave when you do that.

(13) a. John may have left yesterday.
b. ??John can have left yesterday.

(I'm not sure whether (13b) is good on some reading, but it sounds odd and does not have the epistemic interpretation of (13a).)

On the other hand, *'can't'* does have an epistemic reading parallel to *'may not'*:

(14) a. John may not have left yesterday.
b. John cannot have left yesterday.

- Similar preferences/restrictions exist in all language, including Japanese. Often a distinction is lexicalized in one language but not in another (e.g., Fench *pouvoir/savoir*; Russian *umet'/moč'*), although the categories themselves are (presumably) universal. There is much work to be done on the formal classification and typological exploration of these expressions.

1.3 Preview

Day 1. Some basic set theory; standard and modal propositional logic; modal bases and accessibility relations; properties of modal bases; models and frames. Time permitting: Axiom systems and the frames they characterize; First-order modal logic.

Day 2: A close read of Kratzer (1981), still the standard in the formal semantic literature on modality. Time permitting: some problems with the account; syntactic behavior of different classes of modals.

Day 3: Application of modal logic to time; combinations of time and modality; compositional analysis of tense and modality in indicative conditionals (Kaufmann, 2005).

2 Formal preliminaries

2.1 Sets

Sets are collections of objects/individuals/items of any kind.

2.1.1 Some notation

- $\{a, b, c\}$: the set containing a, b, and c
- $\emptyset = \{\}$: the empty set
  (there is only one, that's why it is called *the* empty set)
- $x \in X$: x is an element/a member of X
- $x \notin X$: x is not an element/a member of X

2.1.2 Ways of defining sets

For instance, let O be the set of odd natural numbers.

Enumeration. $O = \{1, 3, 5, \ldots\}$

Membership condition. $O = \{x | x > 0 \text{ and } x \text{ is not divisible by } 2\}$
(notice that the occurrences of x to the right of the bar '|' are bound by the occurrence on the left)

Induction. This works if the set has some internal structure that lets you go through all members systematically.

a. $1 \in O$;

b. if $x \in O$, then $x + 2 \in O$;

c. nothing else is in O.

2.1.3 Things to keep in mind

- Sets are unordered: $\{a, b\} = \{b, a\}$
- Set membership is determined for each object, and it is not a matter of degree: For all x and sets X, either $x \in X$ or $x \notin X$.
  Also: $\{a, b, a\} = \{a, b\}$
- Sets may contain sets: $\{a\} \neq \{\{a\}\}$; $\{a, \{a\}\} \neq \{a\}$.
  Also: $\emptyset \neq \{\emptyset\}$

2.1.4 Relations between sets:

Subset. $X \subseteq Y$: X is a *subset* of Y:
For all x, if $x \in X$ then $x \in Y$

Equality. $X = Y$: X and Y have the same members:
$X \subseteq Y$ and $Y \subseteq X$;
for all x, $x \in X$ if and only if $x \in Y$

Proper subset. $X \subset Y$: X is a *proper subset* of Y:
$X \subseteq Y$ and $Y \not\subseteq X$;
$X \subseteq Y$ and for some x, $x \in Y$ and $x \notin X$

Superset. Superset ($X \supseteq Y$) and proper superset ($X \supset Y$) are defined similarly.

Things to note:

- The empty set is a subset of every set (including itself): For all sets X, $\emptyset \subseteq X$
- Expressions like '$X \subseteq Y$' are *statements about sets* that are either true or false.

2.1.5 Operations on sets

Intersection. $X \cap Y = \{x | x \in X \text{ and } x \in Y\}$

Union. $X \cup Y = \{x | x \in X \text{ or } x \in Y \text{ (or both)}\}$

Relative complement. $X \setminus Y = \{x | x \in X \text{ and } x \notin Y\}$

(You often see the alternative notation '$X - Y$' instead of '$X \setminus Y$'. I may use both in these lectures.)

Complement. This is only defined relative to a given domain of individuals, often called the "universe" and written 'U':

$\overline{X} = U - X$

Things to note:

- Operators like '$\cap$' produce *sets* from sets; this is fundamentally different from the relation symbols above.

Examples:

a. $\{a, b, c, d, e\} \cap \{b, c, 1, 2\} = \{b, c\}$

b. $\{a, b, c, d, e\} \cup \{b, c, 1, 2\} = \{a, b, c, d, e, 1, 2\}$

c. $\{a, b, c, d, e\} - \{b, c, 1, 2\} = \{a, d, e\}$

d. Let $U = \{a, \ldots, z\}$. Then
$\overline{\{a, b, c, d, e\}} = \{f, \ldots, z\}$

2.1.6 Further notions

Cardinality. $|X|$ is the number of the members of X. Notice:

a. $|\{a, b, c\}| = 3$

b. $|\{a, b, c, a\}| = 3$

c. $|\{a, b, c, \{a\}\}| = 4$

d. $|\{a, b, c, \{d, e\}\}| = 4$

e. $|\emptyset| = 0$

f. $|\{\emptyset\}| = 1$

Powerset. $\wp(X)$ is the set of all subsets of X:

$\wp(X) = \{X | Y \subseteq X\}$

Remember that $X \subseteq X$ and $\emptyset \subseteq X$ for any set X! Thus:

$\wp(\{a, b, c\}) = \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \emptyset\}$

Some simple but useful truths:

a. $|\overline{X}| = |U| - |X|$

b. $|X \cup Y| = |X| + |Y| - |X \cap Y|$

c. If $X \subseteq Y$ then $|X| \leq |Y|$

d. $|\wp(X)| = 2^{|X|}$
(e.g., $|\wp(\{a, b, c\})| = 2^3 = 8$)

2.2 Relations

2.2.1 Sequences

Sequences are collections of objects in which (i) order matters and (ii) multiple occurrences of the same object are distinct. Written with angle brackets: '$\langle \ldots \rangle$' Unlike sets (see above):

- $\langle a, b \rangle \neq \langle b, a \rangle$
- $\langle a, b, a \rangle \neq \langle a, b \rangle$

2.2.2 Cartesian Product

(Also called the cross-product) of two sets X and Y:
The set of all pairs whose first member is in X and whose second member is in Y.

$$X \times Y = \{\langle x, y \rangle | x \in X, y \in Y\}$$

- Special case: $X \times X = X^2$
- Useful fact: $|X \times Y| = |X| \cdot |Y|$

2.2.3 Binary relations

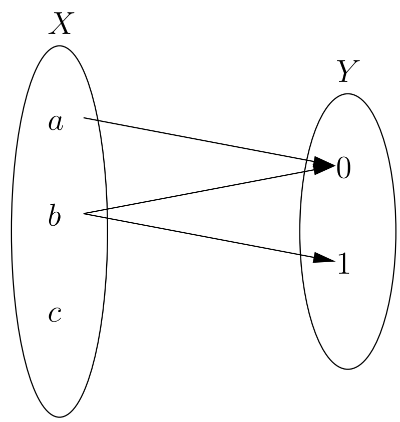
A binary relation R between sets X and Y is a subset of $X \times Y$ (i.e., a set of pairs of members of X and Y).

$$R \subseteq X \times Y$$

- X is the *domain* of R
- Y is the *range* of R
- Example: Let $X = \{a, b, c\}$ and $Y = \{0, 1\}$. Then one relation (among others) with domain X and Y is this:

  $$R = \{\langle a, 0 \rangle, \langle b, 0 \rangle, \langle b, 1 \rangle\}$$

  It can be helpful to represent relations visually, as follows:



- The statement that $\langle x,y\rangle \in R$ is often written as 'xRy'.
- Q: How many relations from X to Y are there?

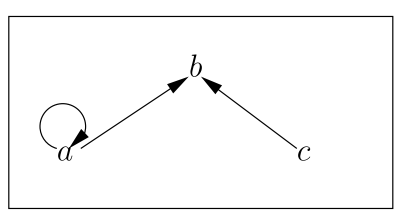
  A: Each such relation is a subset of $X \times Y$, the Cartesian product of X and Y. In other words, each such relation is a member of $\wp(X \times Y)$, the powerset of the Cartesian product of X and Y. Thus there are $|\wp(X \times Y)| = 2^{|X \times Y|} = 2^{|X|\cdot|Y|} = 2^6 = 64$.

  Notice that by the same argument, there are 64 relations from Y to X.
- We will often be looking at relations whose domain and range are the same:

  $R \subseteq X \times X$

  For instance, let X be as before and $R = \{\langle a,b\rangle, \langle a,a\rangle, \langle c,b\rangle\}$

  This can be visualized as follows:



2.2.4 n-ary relations

This is just a generalization of the binary case:

$$\begin{aligned} R^3 \subseteq & X \times Y \times Z \\ = & \{\langle x,y,z\rangle | x \in X, y \in Y, z \in Z\} \end{aligned}$$

2.2.5 Properties of relations

For simplicity, let us consider a binary relation $R \subseteq X \times X$.

serial. For all $x \in X$, there is some $y \in X$ such that xRy.

reflexive. For all $x \in X, \langle x, x\rangle \in R$.

Every object in the domain has a "loop" around it.

non-reflexive. There is at least one $x \in X$ such that $\langle x, x, \rangle \notin R$.

irreflexive. There is *no* $x \in X$ such that $\langle x, x\rangle \in R$.

symmetric. For all $x, y \in X$, if xRy then yRx.

Every arrow is mirrored by an arrow in the opposite direction.

non-symmetric. There is at least one pair $x, y \in X$ such that xRy but not yRx.

antisymmetric. For all $x, y \in X$, if xRy and yRx then $x = y$.

asymmetric. There is no pair $x, y \in X$ such that xRy and yRx.

transitive. For all $x, y, z \in X$, if xRy and yRz then xRz.

non-transitive. There is at least one triple $x, y, z \in X$ such that xRy and yRz but not xRz.

intransitive. For all $x, y, z \in X$, if xRy and yRz, then not xRz.

euclidean. For all $x, y, z \in X$, if xRy and xRz, then yRz.

connected. For all $x, y \in X$, either xRy or yRx or both.

dense. For all $x, y \in X$, if xRy and $x \neq y$, then there is some z such that $x \neq z$ and $y \neq z$ and xRz and zRy.

2.2.6 Classes of relations

equivalence relation. reflexive, symmetric and transitive.

partial order. *weak:* reflexive, transitve, and antisymmetric.

strong: irreflexive, transtivie and asymmetric.

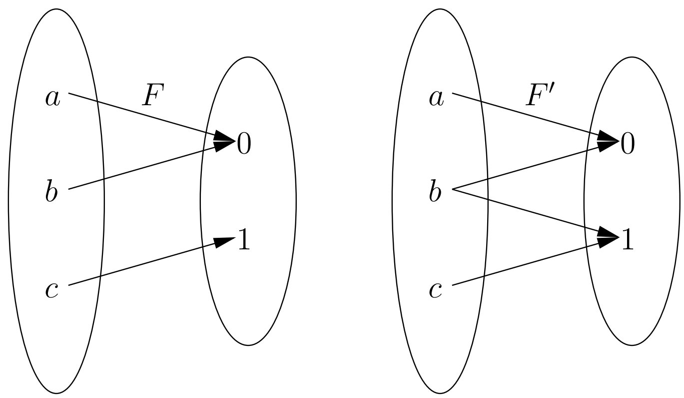
pre-order. reflexive and transitive.

total/linear order. an order that is connected.

3 Functions

- A function F is a relation such that for all x, y, z: If xFy and xFz, then $y = z$.

  Thus in the following picture, F is a function and F' is not:



- A function with domain X and range Y is:
  - *total* if for each $x \in X$, there is some $y \in Y$ such that xFy.
  - *partial* otherwise.
  - By convention, “function” is used to mean “total function.”

- Thus for each $x \in X$, the $y \in Y$ such that xFy (if defined) is *unique*. We usually write ‘$F(x)$’ to refer to this y.

- Notational conventions:
  - $F : X \mapsto Y$ is a function with domain X and range Y.
  - The set of all functions from X to Y is sometimes written ‘Y^X’. Some people find this notation pointless at best, confusing at worst. However, it has one advantage: It reminds us of a way of computing how many (total) functions there are from X to Y: $|Y^X| = |Y|^{|X|}$. Thus in the above pictures, there are $2^3 = 8$ functions from X to Y (out of 64 relations, as you may recall).

4 Propositional logic

Instead of dealing with English, we will start by considering a formal language which is a bit like English in certain relevant respects. It, too, has a syntax and a semantic interpretation, which we will consider separately.

4.1 Syntax

Let a set $\mathcal{A}$ of (atomic) propositional letters be given. The language $\mathcal{L}_\mathcal{A}$ of propositional logic is defined as follows:

a. $\mathcal{A} \subseteq \mathcal{L}_\mathcal{A}$;

b. if $\varphi \in \mathcal{L}_\mathcal{A}$ then $(\neg\varphi) \in \mathcal{L}_\mathcal{A}$;

c. if $\varphi, \psi \in \mathcal{L}_\mathcal{A}$ then $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi) \in \mathcal{L}_\mathcal{A}$;

d. nothing else is in $\mathcal{L}_\mathcal{A}$.

- A more concise way of saying the same thing would be this: "$\mathcal{L}_\mathcal{A}$ is the smallest set containing $\mathcal{A}$ and closed under the unary operator $\neg$ and the binary operators $\wedge$, $\vee$ and $\rightarrow$."

- Parentheses will be omitted when there is no danger of ambiguity. Negation binds more tightly than the binary operators: '$\neg\varphi \wedge \psi$' stands for '$(\neg\varphi) \wedge \psi$'.

- So we get parsetrees for sentences, such as the one in Figure 1.

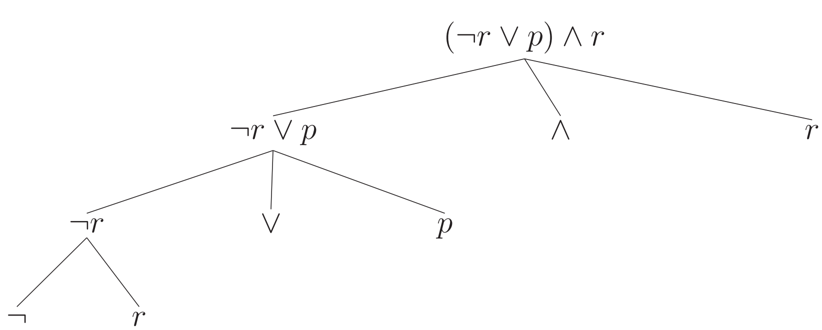


Figure 1: Syntactic analysis of a sentence of propositional logic

φ	ψ	$\neg\varphi$	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$
0	0	1	0	0	1
0	1	1	0	1	1
1	0	0	0	1	0
1	1	0	1	1	1

Figure 2: Truth functions for four sentential connectives

	p	r	$\neg r$	$\neg r \vee p$	$(\neg r \vee p) \wedge r$
V_1	0	0	1	1	0
V_2	1	0	1	1	0
V_3	0	1	0	0	0
V_4	1	1	0	1	1

Figure 3: Truth table for the sentence in Figure 1

4.2 Semantics

Sentences are interpreted with respect to *valuation functions* (or truth assignments). Each valuation function V fixes the truth values of the atomic sentences; the values of the compound sentences are determined (compositionally) by the truth values of their constituents and the connectives that combine them.

Formally, each connective corresponds a *truth function* in $\{0, 1\}^{\mathcal{A}}$, i.e., a function from atomic sentences to of the numbers 0 ("false") and 1 ("true"). The truth functions for the four connectives we introduced above are given in Table 2.

In this table, φ and ψ are meta-variables ranging over atomic or complex sentences. Truth values for complex sentences are defined *recursively* in terms of these tables. For the example sentence in Figure 1, this is spelled out in Table 3.

Recall that there are $|Y|^{|X|}$ functions with domain X and range Y. Thus in particular, there are 2^n distinct (total) assignments of truth values to n atomic sentences. In the example sentence in Table 3, there are two atomic sentences, p and r, and correspondingly four distinct truth assignments. In the table, they are called 'V_1' through 'V_4'. As we see, this sentence is true case if and only if $V(p) = V(r) = 1$. In the table this is only the case with V_4.

It is useful to pause for a moment and consider what these four truth functions represent. In some intuitive sense, they correspond to "states of affairs." What this means becomes clear if we consider English sentences

	p	q	r
V_{1a}	0	0	0
V_{1b}	0	1	0
V_{2a}	1	0	0
V_{2b}	1	1	0
V_{3a}	0	0	1
V_{3b}	0	1	1
V_{4a}	1	0	1
V_{4b}	1	1	1

Figure 4: Eight truth functions with three atomic sentences

instead of p and r, and the English expressions *'it is not the case that'* for $\neg$ and *'and'* for $\wedge$. For instance, consider the English sentence in (15). (We use brackets to indicate the intended syntactic structure—English, unlike the language of propositional logic, is ambiguous.)

(15) [Either Mary doesn't smile or Bill is happy], and Mary smiles.
  a. p = Bill is happy
  b. r = Mary smiles

Each of (15a,b) is true in two cases: (15a) is true if Bill is happy and Mary smiles, and if Bill is happy and Mary doesn't smile. Likewise, (15b) is true if Mary smiles and Bill is happy, and if Mary smiles and Bill is unhappy.

Clearly, the number of cases in which these sentences are true depends on the number of atomic sentences in our language. If there is a third sentence in $\mathcal{A}$ (say, q = *'Today is Friday'*), we could distinguish between *eight* different truth assignments ($= 2^3$), not just four. Each of (15a,b) is then true in four of the eight, not just in two. Figure 4 lists the eight possibilities.

Intuitively, with the addition of the new (logically independent) sentence q, we get a more "fine-grained" view of the world. V_1 captured a certain state of affairs, but a closer look reveals that this state of affairs really comprises two different cases: V_{1a} and V_{1b}. Thus in a way, each of V_{1a} and V_{1b} is more informative than V_1. Put differently, V_1 does not tell us the whole story—it gives us only *partial* information about the facts. With more sentences, we get more distinctive power. And there is no reason to put an upper limit on the number of atomic sentences. So what are the (hypothetical) endpoints of this process of refinement? What are the situations that give truth values to *all* atomic sentences (and therefore all compound sentences in the language)? Those are *possible worlds*, the notion at the center of modal logic.

5 Propositional modal logic

5.1 Possible worlds

§1 History: can be traced back to Leibniz, who believed that we live in a world which is one of infinitely many possible worlds created by God (and, fortunately for us, the best one among them!).

§2 In philosophical logic, the notion was introduced only in the 1960s, independently by Hintikka (1961) and Kripke (1963).

§3 Ontological status continues to be controversial among philosophers, but these debates will not concern us here. The utility of possible worlds as a methodological tool in semantic analysis does not depend on any particular stance on metaphysical questions, such as whether worlds other than ours "exist" in any real sense. For our purposes, they are nothing but abstract entities which help us in modeling certain semantic relations between linguistic expressions.

§4 Minimal assumption: Possible worlds fix the denotations of the relevant expressions — truth values for sentences, properties for verb phrases, and so on. For now, we will limit the discussion to sentences. Furthermore, we will continue to use the language of standard propositional logic, as before.

§5 The meaning of a sentence is analyzed in terms of its role in distinguishing between possible worlds. Assuming that every sentence is guaranteed to be either true or false (the Law of Excluded Middle)[2] but not both true and false (the Law of Non-contradiction), each possible world determines the truth values of all atomic sentences and via the interpretation of the logical connectives, those of their Boolean combinations.

5.2 Models

§1 We have already encountered the notion of truth assignments. In the previous section, an assignment was a function $V : \mathcal{L}_\mathcal{A} \mapsto \{0, 1\}$. Each "world" corresponded to a different assignment function V. We will continue along these lines; formally, however, we'll have *one* function $V : W \mapsto (\mathcal{L}_\mathcal{A} \mapsto \{0, 1\})$ from worlds to truth assignments.

[2] This is a simplification. Sentences carrying semantic presuppositions are commonly assumed to be neither true nor false at worlds at which their presuppositions are not satisfied.

§2 *First step:* A *model* is a pair $M = \langle W, V\rangle$, consisting of a non-empty set W (the set of possible worlds) and a function V which, for each world w in W, assigns truth values to the sentences in the language, satisfying the following constraints:

(16) For atomic p, $V_w(p)$ is either 1 or 0

$$V_w(\neg\varphi) = \begin{cases} 1 \text{ if } V_w(\varphi) = 0 \\ 0 \text{ otherwise} \end{cases}$$

$$V_w(\varphi \wedge \psi) = \begin{cases} 1 \text{ if } V_w(\varphi) = 1 \text{ and } V_w(\psi) = 1 \\ 0 \text{ otherwise} \end{cases}$$

$$V_w(\varphi \vee \psi) = \begin{cases} 1 \text{ if } V_w(\varphi) = 1 \text{ or } V_w(\psi) = 1 \\ 0 \text{ otherwise} \end{cases}$$

$$V_w(\varphi \rightarrow \psi) = \begin{cases} 1 \text{ if } V_w(\varphi) = 0 \text{ or } V_w(\psi) = 1 \\ 0 \text{ otherwise} \end{cases}$$

These clauses are just a different notation for the truth functions given in the Table in Figure 3 above.

§3 Given the assignment function, each sentence in the language distinguishes between those worlds in which it is true and those in which it is false. We may thus associate each sentence φ with the set of those worlds in which it is true. To this end, we introduce a function $[\![\cdot]\!]^M$ for our model M, mapping sentences to sets of worlds:

(17) $$[\![\varphi]\!]^M =_{\text{def}} \{w \in W | V_w(\varphi) = 1\}$$

We will drop the superscript '^{M}' where no confusion can arise.

§4 We call $[\![\varphi]\!]^M$ the *denotation* of sentence φ in M. The term *proposition*, in its technical use in this context, is reserved for sets of worlds. Thus the denotation of the sentence *'it is raining'* is the proposition "that it is raining," the set of just those worlds in which it is raining.

5.3 Semantic relations

§1 Now we can characterize standard logical properties of sentences set-theoretically in terms of the propositions they denote:

(18) A sentence φ is:
  a. *a tautology (tautologous)* iff $W \subseteq [\![\varphi]\!]$

b. *a contradition (contradictory)* iff $[\![\varphi]\!] \subseteq \emptyset$
c. *a contingency (contingent)* otherwise

§2 Likewise, semantic relationships between sentences can be defined in terms of the propositions they denote. For instance, the interpretation of our logical connectives gives rise to the following relations between the denotations of complex sentences and the parts they are composed of:

(19)
$$\begin{aligned} [\![\neg\varphi]\!] &= W \setminus [\![\varphi]\!] \\ [\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] \\ [\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \cup [\![\psi]\!] \\ [\![\varphi \rightarrow \psi]\!] &= W \setminus ([\![\varphi]\!] \setminus [\![\psi]\!]) \end{aligned}$$

For instance, the clause for conjunction states that the denotation of *'it is raining and it is cold'* is the intersection of the denotations of *'it is raining'* and *'it is cold'*, i.e., the set of worlds in which both of these sentences are true.

§3 This idea can be generalized to arbitrary sets Φ of propositions, writing $[\![\Phi]\!]$ for the set of worlds in which all sentences in Φ are true — i.e., the intersection of their respective denotations:[3]

(20) $[\![\Phi]\!]^M =_{\text{def}} \bigcap\{[\![\varphi]\!]^M | \varphi \in \Phi\}$

§4 Using this notation, we define the the following notions, which are at the center of modal logic:

(21) **Consistency:**
a. A set Φ of sentences is consistent iff there is some world in which all sentences in Φ are true;
i.e., iff $[\![\Phi]\!] \neq \emptyset$.
b. A sentence φ is consistent with a set Φ of sentences iff it can be added to Φ consistently;
i.e., iff $\Phi \cup \{\varphi\}$ is consistent;
i.e., iff φ is true at some world in $[\![\Phi]\!]$.

(22) **Consequence:** A sentence φ is a consequence of a set Φ of sentences if and only if φ is true in all possible worlds in which all sentences in Φ are true;

[3] One could think of $[\![\Phi]\!]^M$ as the denotation of the conjunction of all members of Φ, but this analogy breaks down if Φ has infitely many members. In this case, $[\![\Phi]\!]^M$ is still defined, but our language does not include the corresponding infinitary conjunction.

i.e., iff φ is true at all worlds in $[\![\Phi]\!]$.

§5 In what follows, we will sometimes use these terms relative to just a single proposition. In a slight abuse of terminology, we will speak of a sentence being consistent with, or a consequence of, such a single proposition, rather than the singleton set containing it. This is safe, for notice that $[\![\{\psi\}]\!] = [\![\psi]\!]$.

5.4 Modal base and modal force

Everything in the area of modality is about the relations of consistency and consequence between a sentence φ (the sentence embedded under the modal) and some underlying background or body of information (called 'A' in the introduction). This background information, relative to which modals are interpreted, is generally represented as a set of possible worlds. The expression '$[\![\Phi]\!]$' used refers to objects of this type.

There are two different but interdefinable ways of encoding this. In modal logic, it is given as an *accessibility relation*; in the linguistic tradition that started with Kratzer, it is known as a *modal base*. Both of these perspectives are widely used, and there is a straightforward relationship between them. Anyone who works in this area should be familiar with both.

§1 Above, we defined the notions of consistency and consequence relative to the proposition $[\![\Phi]\!]$, the set of those worlds at which all sentences in Φ are true. Following Kratzer (1979, 1981), we call this set of worlds the *modal base*.[4]

§2 Modal bases serve to distinguish between the different readings of modals (epistemic, deontic, etc.) we identified in the introduction, thus we speak of epistemic, deontic, and other modal bases. The intuition is that in each case the modal base is the set of just those worlds that are compatible with all of the speaker's beliefs, desires, the applicable laws, and so on.

§3 *World dependence.* The contents of speakers' beliefs, laws etc. are of course themselves contingent, i.e., they may vary from world to world. Thus we cannot expect the contents of, say, an epistemic modal base to remain constant across different worlds: Just like the truth values of the sentences in our language, the modal base depends on the world of evaluation.

[4]The set of propositions Φ whose intersection forms the modal base is called *conversational background* by Kratzer. For simplicity, I sidestep the latter in this section, giving a more direct definition of modal bases.

§4 *Formal representation.* A modal base is given by a function $R : W \mapsto (W \mapsto \{0, 1\})$, i.e., from possible worlds to (characteristic functions of) propositions. We will reserve the letter R as a general symbol for such functions, using superscripts to distinguish between them, such as R^{epist} and R^{deont} for epistemic and deontic modal bases, respectively. Given a world w of evaluation, we may write R_w^{epist} for the (speaker's) epistemic modal base at w, i.e., the set of just those worlds that are compatible with what the speaker's beliefs.

§5 *Modal force.* Aside from modal bases, the second major parameter in the interpretation of modality is the *modal force.* Unlike the modal base, which is usually left implicit and contextually given, the modal force is an integral part of the lexical meaning of all modals.[5] For instance, both (23b) and (23c) can be used as assertions about either the speaker's beliefs ("given what I know…") or John's obligations ("given the laws and John's age…"), but the logical relations invoked in the two sentences are unambiguously possibility and necessity, respectively. With respect to the intended modal base, this difference between (23b) and (23c) corresponds to consistency and consequence.

(23) a. John is at the party.
b. John may be at the party.
c. John must be at the party.

§6 *Truth conditions for (23b,c).* We assume that in both cases the modal auxiliary functions as a sentential operator which takes (23a) as its argument, and we assume that the intended modal base ρ is given. We extend our formal language with symbols for these operators, $\Diamond$ and $\Box$, respectively: For all sentences φ in $\mathcal{L}_{\mathcal{A}}$ and modal bases R, $\Diamond_R\varphi$ and $\Box_R\varphi$ are also in $\mathcal{L}_{\mathcal{A}}$. The valuation function V is extended accordingly:

(24) $$V_w(\Diamond_R\, \varphi) = \begin{cases} 1 \text{ if } V_{w'}(\varphi) = 1 \text{ for some } w' \in \rho_w \\ 0 \text{ otherwise} \end{cases}$$

(25) $$V_w(\Box_R\, \varphi) = \begin{cases} 1 \text{ if } V_{w'}(\varphi) = 1 \text{ for all } w' \in \rho_w \\ 0 \text{ otherwise} \end{cases}$$

§7 Usually, only one of these operators is taken as basic; the other is then defined in terms of it. This can be one either way:

(26) a. $\Diamond\varphi =_{\text{def}} \neg\Box\neg\varphi$

[5]This is not to say that the modal force is always clear-cut. It can be (and usually is) vague.

b. $\Box\varphi =_{\text{def}} \neg\Diamond\neg\varphi$

Most authors choose (26a).

§8 Remember that the definition in (24) is a shortcut compared to Kratzer's treatment. She defines the modal base indirectly via a "conversational background," a function from worlds to sets of propositions. Thus for instance, writing CG^{epist} for an epistemic conversational background, we have $R^{epist}_w = [\![CG^{epist}_w]\!]$ for all worlds w. The more direct definition here is not meant to deny the utility of conversational backgrounds; we'll see more on it later.

§9 *Accessibility relations* This takes us to the area of contact between the Kratzer-style theory and the modal-logical approach in terms of *accessibility relations* between possible worlds. For any function ρ from worlds to sets of worlds, there is a relation R_ρ which pairs up each world w with all and only the worlds in ρ_w:[6]

(27) $R_\rho =_{\text{def}} \{\langle w, w'\rangle | w' \in \rho_w\}$

Many authors, especially in philosophical logic, take such relations R between possible worlds as basic, rather than defining them in terms of modal bases.

§10 *Models.* Typically, modal bases are given as part of the model. Assuming that we are only interested in one modal base R, model is a triple $\langle W, R, V\rangle$, where W and V are as before and R is an accessibility relation.

§11 *Frames.* The pair $\langle W, R\rangle$ — the set of worlds and the accessibility relation — is called a *frame.* For a fixed frame, certain inference patterns may be guaranteed to hold solely in virtue of the structure of the modal base, regardless of what truth values V assigns to the sentences of the language at individual worlds. We will discuss some of them later. In this discussion, it will be useful to switch back and forth between modal bases as functions from worlds to propositions and as accessibility relations.

[6] Every function $F : A \mapsto (B \mapsto C)$ corresponds to a unique function in $F' : (A \times B) \mapsto C$, in the sense that for all $a \in A, b \in B$ and $c \in C$, $(F(a))(b) = c$ iff $F'(a, b) = c$. The switch from one representation to the other is known to some as "Schönfinkelization," and to others as "Currying," both named after famous logicians.

In our case, a modal base is a function from worlds to (characteristic functions of) sets of worlds, whereas an accessibility relation is (the characteristic function of) a set of pairs of worlds, i.e., a relation between worlds.

5.5 Properties of frames/modal bases

1. Consistency
2. Realism and total realism
3. Positive and negative introspection

The modal base is not only a useful parameter to capture the variability and context dependence of modal expressions. It is also the right place to state generalizations about the properties of particular readings of modals. The investigation of such general properties and their logical consequences is the topic of modal logic; introductions into this field (Hughes and Cresswell, 1996, etc.) go far beyond what we can cover here.

For concreteness, unless otherwise stated, we will restrict the discussion in this section to the special case of a *doxastic* modal base — i.e., one that represents a speaker's beliefs. We will write 'R' instead of R^{dox} for simplicity.

5.5.1 Consistency.

§1 Formally, this corresponds to the requirement that for all worlds w, the set R_w be non-empty.

§2 Alternatively, in relational terms, the requirement is that R be *serial*: For every world w, there is at least one world w' such that wRw'.

§3 Kratzer (1981) uses the term "normal" for consistent modal bases.

§4 In linguistic theory, consistency is generally taken to be a requirement for all modal bases. An inconsistent modal base may result in the interpretation of sentences by quantification over an empty set (of worlds), leading to presupposition failure. This is part of a more general assumption that (most) quantifying expressions in natural language presuppose that the domain of quantification is not empty.

5.5.2 Realism.

Next, one may impose the condition that a modal base be *realistic*, in the sense that none of the sentences supported by the modal base for a world w (i.e., true at all worlds in R_w) are false at w.

§1 Formally: each world w is a member of R_w.

§2 In relational terms: R is *reflexive* — i.e., for each world w, wRw.

§3 Realism is a sensible condition for some modalities, but not for others. In terms of speakers' beliefs, the condition means that all of the speaker's beliefs are true. This property is often assumed to distinguish *knowledge* from *belief*, or equivalently, *epistemic* from *doxastic* accessibility relations. Deontic modality is generally not realistic. Assuming otherwise would amount to the claim that all obligations are fulfilled.[7]

Total realism.

Kratzer (1981) introduces this special kind of realism, which she uses mostly in interpreting counterfactual conditionals. A modal base is totally realistic iff it identifies the world of evaluation uniquely.

§1 Formally: for each world w, $R_w = \{w\}$.

§2 In terms of relations: R is the *identity* relation: wRw' iff $w = w'$.

§3 For comparison:

Realism	Total realism
$w = w' \Rightarrow wRw'$	$w = w' \Leftrightarrow wRw'$
$\Box_R\varphi \Rightarrow \varphi$	$\Box_R\varphi \Leftrightarrow \varphi$

Digression on realism. von Stechow (2004) treats the difference between realistic and non-realistic modal bases as one of the main taxonomical criteria for modals. Here is his list:

(28) Realistic:
  a. epistemic (what is known)
  b. circumstantial (the relevant facts/circumstances)
  c. dispositional (the "structure" of the subject (John, the radio, etc.))
  d. physical (the natural laws)

(29) Non-realistic:
  a. deontic (what the law prescribes, God's will, etc.)
  b. doxastic (what I believe, what they say ...)
  c. teleological (the goals, the strategy)
  d. buletic (what I want, what my mother wants)
  e. stereotypical (the normal course of events)

But this is just one proposal among others. There is still much debate around the dividing lines. I would disagree with von Stechow's use of the

[7]Notice that the absence of the condition does not mean that w *cannot* be a member of R_w. It merely does not have to be.

term "doxastic" to refer to evidentials, in light of the different use of the term in philosophy and some writings on semantics (e.g., Stalnaker, 2002). *End of digression.*

5.5.3 Introspection

Under a doxastic interpretation, R_w is the set of those worlds that are compatible with what the speaker believes at w. These worlds will differ among each other with respect to the truth values of non-modal sentences, such as '*it is raining*'. In addition, they may also differ with respect to the speaker's beliefs: Each world w' in R_w is one in which the speaker has certain beliefs, represented by the set $R_{w'}$. This means for w that the modal base does not only encode the speaker's beliefs about the facts, but also her beliefs about her beliefs.

Clearly some conditions ought to be imposed on these beliefs about one's own beliefs. Suppose the speaker believes in w that it is raining, and suppose further that there are two worlds w', w'' in R_w such that in w' the speaker beliefs that it is raining, and in w'' she believes that it is not. While there is nothing wrong with this scenario from a formal point of view, it is very peculiar indeed as a representation of an agent's beliefs: It amounts to the claim that the speaker has a very definte opinion on the question of whether it is raining, but does not know what that opinion is!

Two conditions that are commonly imposed to avoid such peculiar consequences are positive and negative introspection.

Positive introspection.

§1 This is the requirement that at each world compatible with what the speaker believes (i.e., each world in R_w), she has all the beliefs that she actually has at w (and possibly more).

§2 Formally: for each belief-world w', the speaker's doxastic modal base is a subset of the actual modal base R_w:
For all $w' \in R_w$, $R_{w'} \subseteq R_w$

§3 Relationally: R is *transitive*:
If wRw' and $w'Rw''$, then wRw''

§4 It is helpful to visualize this constraint by illustrating the kind of case it rules out. Such a case is shown in Figure 5. The modal bases R_w and $R_{w'}$ are indicated as partial spheres (whether the worlds are in their respective modal bases, and whether there is any overlap between R_w and $R_{w'}$, is not relevant here). Positive introspection fails because $R_{w'}$

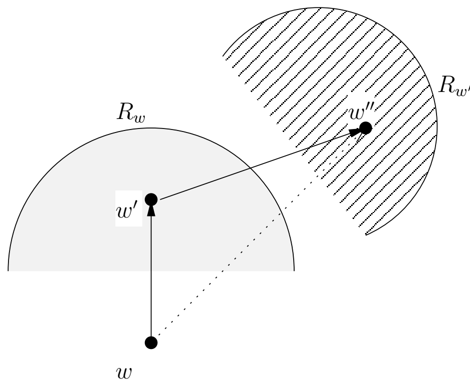


Figure 5: A violation of transitivity: No direct accessibility link leads from w to w'', and $R_{w'}$ is not a subset of R_w.

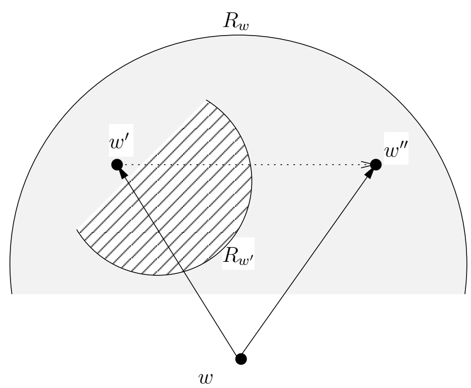


Figure 6: A violation of euclidity: No direct accessibility link leads from w' to w'' (or vice versa), and R_w is not a subset of $R_{w'}$.

is not fully contained in R_w: There is at least one world, w'', which is in $R_{w'}$ but not in R_w.
Equivalently, in terms of the accessibility relation, there is a path leading from w to w'', but w'' is not directly accessible from w.

§5 Positive introspection is usually imposed as a condition on epistemic and doxastic modal bases.

Negative introspection.

§1 This is the corresponding requirement that there be no world compatible with what the speaker believes at which she does holds any beliefs

Table 2: Some properties of modal bases and accessibility relations. Free variables w, w', w'' are assumed to be universally quantified over.

Modal Base	*Accessibility Relation*	*Axiom*	
consistency	seriality	$\Box_R\varphi \Rightarrow \Diamond_R\varphi$	(D)
$R_w \neq \emptyset$	$\exists w'.wRw'$		
realism	reflexivity	$\Box_R\varphi \Rightarrow \varphi$	(T)
$w \in R_w$	wRw		
total realism	identity	$\Box_R\varphi \Leftrightarrow \varphi$	
$R_w = \{w\}$	$wRw' \Leftrightarrow w = w'$		
positive introspection	transitivity	$\Box_R\varphi \Rightarrow \Box_R\Box_R\varphi$	(4)
$w' \in R_w \Rightarrow R_{w'} \subseteq R_w$	$wRw' \wedge w'Rw'' \Rightarrow wRw''$		
negative introspection	euclidity	$\Diamond_R\varphi \Rightarrow \Box_R\Diamond_R\varphi$	(5)
$w' \in R_w \Rightarrow R_w \subseteq R_{w'}$	$wRw' \wedge wRw'' \Rightarrow w'Rw''$		

that she does not actually hold.

§2 Formally: For all $w' \in R_w$, $R_w \subseteq R_{w'}$

§3 Relationally: R is *euclidean*:
If wRw' and wRw'', then $w'Rw''$

§4 As before, it is helpful to give an example in which this condition is violated. The accessibility relation in Figure 6 is not euclidean, and the modal base R_w lacks the property of negative introspection. Even though w' and w'' are both accessible from w, w'' is not accessible from w'.

§5 Negative introspection (i.e., euclidity) is usually required of epistemic and doxastic modal bases.

5.6 Interim summary

Table 2 summarizes these properties of modal bases and the corresponding properties of accessbility relations, along with the axiom of modal logic that is guaranteed to hold for any modal base with the respective properties.

In most applications, it is some combination of these conditions that in its totality determines the properties of a particular modal base and distinguishes it from others. For instance, *doxastic* modal bases (those corresponding to beliefs) are generally taken to be consistent and fully introspective, whereas *epistemic* ones (modeling knowledge, rather than mere belief) are, in addition to these properties, realistic. Realism, as a property of sentential operators, corresponds to *veridicality*, which is generally taken to distinguish knowledge from belief. For deontic modal bases, on the other hand, realism should not be postulated as a general property, since we cannot infer from the fact that something ought to be the case that it is.

Positive or negative introspection are rarely applied to modal bases other than epistemic or doxastic ones. One example of such an application is von Stechow's (2004) claim that the modal base corresponding to one's abilities should be negatively introspective:

> An *ability background* is a function H_F which maps a [pair consisting of an] individual and a situation to a set of situations s' such that x has all abilities in s' that x has in s.

6 Some Logic

Before moving on, we will relate the discussion so far to the literature on modal logic. An important concern in that field, which is relevant for our purposes, is the relationship between logical *systems* and the *frames* that correspond to them.

§1 A "system" here is a "logic" in the philosophical sense. It consists of (i) a set of axioms (sentences that are assumed to be true) and (ii) rules for deriving new sentences from given ones.

§2 A sentence φ is a *theorem* of the system if and only there is a *proof* for it which relies only on the axioms, and rules of the system (and other theorems).[8]

§3 Semantically, sentences are interpreted with the way we've seen before:

- A *frame* is a pair $\langle W, R\rangle$, where W is a non-empty set and R is a relation in $W \times W$.
- A *model* is a triple $\langle W, R, V\rangle$, where $\langle W, R\rangle$ is a frame and V is a truth assignment to sentences (pointwise for members of W).
- We call W the set of *possible worlds* and R the *accessibility relation.*

§4 *Truth: Assigned at worlds in a model.*
Given a model $\langle W, R, V\rangle$, a sentence φ is *true* at $w \in W$ iff $V_w(\varphi) = 1$.

§5 *Validity in a model: Truth at all worlds.*
A sentence is true in a model $\langle W, R, V\rangle$ iff it is true at all $w \in W$.

§6 *Validity on a frame: Validity in all models.*
A sentence is valid on a frame $\langle W, R\rangle$ iff it is valid in $\langle W, R, V\rangle$ for all truth assignments V.

§7 Now there are two ways in which one sentence ψ can "follow from" a sentence φ:

- syntactically: ψ can be derived as a theorem from the axioms together with φ; or
- semantically: ψ is true whenever φ is true (i.e., in all models at all worlds at which φ is true).

[8] A proof is a finite sequence Γ of sentences such that (i) each sentence in Γ is either an axiom or derived by application of the rules from sentences earlier in Γ, and (ii) φ is in Γ.

§8 It is quite clear that syntactically, which sentences can be derived from others depends on the set of axioms one starts out with. Likewise, which sentences are guaranteed to be true whenever other sentences are true depends on the class of models one considers.

Soundness: A system S (of axioms and rules) is *sound* with respect to a class of models iff all the theorems derivable in S are valid for the models in that class.

Completeness: A system S is *complete* with respect to a class of models iff all the sentences that are valid in that class of models are theorems of S.

§9 Much of the work in modal logic is spent exploring the relationship between axiomatic systems on the one hand and classes of models, on the other.
The names of commonly used systems follow a somewhat arcane and unsystematic schema. Below are a few of them: K, T, D, S4, S5, and KD45.

6.1 System K

"K" stands for the name of Saul Kripke, whose work in the 1960s was instrumental in developing the formal methods of modal logic used today.[9]

(PC) If α is a tautology of propositional calculus, then α is an axiom of K.

(K) $\Box(p \to q) \to (\Box p \to \Box q)$

- Strictly speaking, **PC** is an *axiom schema*, giving us infinitely concrete axioms; one for each propositional tautology, such as $p \vee \neg p$ or $(p \to q) \vee q$. Clearly **PC** is valid on all frames, since tautologies are true at all worlds by definition, and their truth does not depend on the accessibility relation.

- **K** is also valid on all frames. Can you convince yourself that this is the case, and that it is intuitively correct?

Together with **PC** and **K**, system K has the following rules:

[9] I follow the convention in Hughes and Cresswell (1996), writing the names of systems in normal font and those of axioms in boldface. Notice that HG write '$\supset$' for the material conditional (here written '$\to$') and 'L' and 'M' for the modal operators $\Box$ and $\Diamond$.

(US) If α is a theorem, then so is $\alpha[\beta_1/p_1, \ldots, \beta_n/p_n]$ (the result of substituting the sentence β_i for all occurrences of the atomic propositional variable p_i in α).

(MP) If α and $\alpha \to \beta$ are theorems, then so is β.

(N) If α is a theorem, then so is $\Box\alpha$.

- (US), (MP) and (N) stand for "uniform substitution," "modus ponens," and "necessitation," respectively.
- (US) merely states that the logical properties of sentences depend only on their form, not on the particular expressions used in them. (MP) is an intuitively plausible statement of the role of the material conditional in inference.

  (N) may seem odd at first, but remember that it is a statement about validity, not truth: The claim is *not* that p entails $\Box p$, but rather that if p is valid (true at all worlds in all models of a certain class), then so is $\Box p$. Stated this way, it should be obvious.

With the three rules of derivation, **PC** and **K** are valid on *all* frames. So these are really absolutely fundamental building blocks of modal logic and possible-worlds semantics. All systems of modal logic that lend themselves to an interpretation in terms of possible-worlds semantics can be expressed as "K+X" or simply "KX," where "X" is a series of axioms added to **PC** and **K**. However, some common systems have names that are not transparent in this way.

6.2 System T

This could also be called "KT"; the added axiom is the following:

(T) $\Box p \to p$

As mentioned above, there is a connection between collections of axioms on the one side and classes of frames on the other. We can use **T** as a simple example to illustrate this relationship. As it happens, **T** is valid on all and only the *reflexive* frames—i.e., those frames $\langle W, R\rangle$ such that for all $w \in W$, wRw. How can we show this? Here is an informal sketch of the proof.

Fact 1
T *is valid on a frame* $\langle W, R\rangle$ *if and only if* R *is reflexive.*

Proof. The claim is a biconditional: **T** is valid on $\langle W, R \rangle \Leftrightarrow R$ is reflexive. We prove both directions in turn by *reductio ad absurdum*. (There are other ways of doing it as well.)

($\Rightarrow$) Suppose **T** is valid on a frame $\langle W, R \rangle$ that is not reflexive. Thus there is some world $w \in W$ such that not wRw. Consider a truth assignment V for this frame with the following properties: For some sentence p, (i) $V_w(p) = 0$ and (ii) for all w' such that wRw', $V_{w'}(p) = 1$. By (ii), $V_w(\Box p) = 1$; by (i), $V_w(p) = 0$. So $V_w(\Box p \to p) = 0$, contrary to the assumption that **T** is valid.

($\Leftarrow$) Let $\langle W, R \rangle$ be a reflexive frame on which **T** is not valid. Thus there is an assignment V for this frame such that for some world $w \in W$ and some sentence p, $V_w(\Box p \to p) = 0$—that is, (i) $V_w(\Box p) = 1$ and (ii) $V_w(p) = 0$. From (i), it follows that there is no world w' such that wRw' and $V_{w'}(p) = 0$. But since p is false at w, this means that not wRw, contrary to the assumption that R is reflexive. $\Box$

System T in itself has no special place in linguistic theory. However, axiom **T** is important in distinguishing some systems from others. In words, **T** says "If φ is necessary, it is true." Whether this is a desirable property depends on the intended meaning of "necessity."

In terms of natural-language sentential operators, **T** corresponds to the property of *veridicality*: Op(φ) entails φ. For instance, veridicality is commonly assumed for knowledge, but not belief.

6.3 System D

This is obtained by adding to K the following axiom:

(D) $\Box p \to \Diamond p$

Thus the system is really "KD," but it is known as "D" because of its close association with *deontic* logic. Here the necessity operator is interpreted as "it is required that p"; authors generally assume that this should entail "it is permissible that p." But none of this is set in stone—whether this should be an axiom depends on whether you want to account for contradictory requirements, something that certainly can happen in real life.

As before, we have a correspondence between the axiom and the class of frames on which it is valid:

Fact 2
D *is valid on a frame* $\langle W, R \rangle$ *iff* R *is serial.*

A relation is serial iff for each $w \in W$, there is some $w' \in W$ such that wRw'. The proof of the fact is left as an exercise.

Relationship between T and D. There is a one-way relationship between the systems T and D: Axiom **D** is a theorem of T, but **T** is not a theorem of D. This fact is not immediately obvious; the first of these claims is easy to prove:

(30)	a.	$\Box p \to p$	(**T**)
	b.	$\neg p \to \neg\Box p$	(a., Contraposition)
	c.	$\neg p \to \Diamond\neg p$	(b., Def. of $\Diamond$)
	d.	$p \to \Diamond p$	(c., $[\neg p/p]$)
	e.	$\Box p \to \Diamond p$	(a. and d., Syllogism)

The second claim is quite a bit harder to prove formally, but it is easy to see intuitively why it must be true. Recall that T characterizes the class of all reflexive frames and D characterizes the class of all serial frames. Now clearly all reflexive frames are serial, but not all serial frames are reflexive. (I.e., there may be worlds with links to other worlds but not to themselves.) And indeed, there are frames that are serial (and on which **D** is valid) but not reflexive (so **T** is not valid on them).

6.4 System S4

S4 follows a different naming scheme.[10] In our scheme, it would be called "KT4." Thus it has the axioms and rules of K, T, and a new rule called "4":

(**4**) $\Box p \to \Box\Box p$

Fact 3
__4__ is valid on a frame $\langle W, R\rangle$ iff R is transitive.

Proof. ($\Rightarrow$) Suppose **4** is valid on a frame $\langle W, R\rangle$ that is not transitive. Thus there are worlds $w, w', w'' \in W$ such that wRw' and $w'Rw''$, but not wRw''. Consider an assignment V such that $V_w(\Box p) = 1$ (thus in particular, $V_{w'}(p) = 1$) and $V_{w''}(p) = 0$. Then clearly $V_w(\Box p) = 1$ and $V_w(\Box\Box p) = 0$, contrary to the assumption that **4** is valid on the frame.

($\Leftarrow$) Suppose **4** is not valid on a transitive frame $\langle W, R\rangle$. For **4** to be invalid, there must be a counterexample: a world w, an assignment V and

[10] Hughes and Cresswell explain that the names S4 and S5 were first used by Lewis and Langford (1932).

a sentence p such that $V_w(\Box p) = 1$ and $V_w(\Box\Box p) = 0$. Notice that for the latter, there have to be two worlds w', w'' such that wRw' and $w'Rw''$ (otherwise $V_w(\Box\Box p)$ would be vacuously true) and such that $V_{w''}(p) = 0$. But since the frame transitive, we wRw'', thus $V_w(\Box p) = 0$ and w is not a counterexample to **4** after all. $\Box$

Notice that **4** is valid on all transitive frames, but S4 (since it also contains **T**) characterizes all frames that are reflexive and transitive. In out alternative terminology, this is class of modal bases that are *realistic* and *positively introspective*.

6.5 System S5

Parallel to S4, S5 could be called "KT5."[11] It is system K with **T** and the following:

(5) $\Diamond p \to \Box\Diamond p$

Fact 4
***5** is valid on a frame $\langle W, R\rangle$ iff R is euclidean.*

Proof. Omitted. $\Box$

An accessbility relation is euclidean if and only if for all w, w', w'', if wRw' and wRw'' then $w'Rw''$.

Thus S5 characterizes the class of frames that are *reflexive* and *euclidean*. These frames have the following additional properties:

Fact 5
If a relation is reflexive and euclidean, it is also symmetric and transitive.

Proof. *Symmetry.* Consider w, w' such that wRw'. By reflexivity, wRw. Thus wRw' and wRw, so by euclidity, $w'Rw$.

Transitive. Consider w, w', w'' such that wRw' and $w'Rw''$. By symmetry (earlier result), $w''Rw'$ and $w'Rw$. Thus $w'Rw$ and $w'Rw''$, so by euclidity, wRw''. $\Box$

[11] Even more confusingly, Hughes and Cresswell give it the alternative name "KTE," using an alternative label for axiom **5**, **E**, which they attribute to Lemmon and Scott (1977).

Recall that a relation that is reflexive, symmetric and transitive is an *equivalence relation*. Thus S5 characterizes the class of *equivalence frames*—ones in which the accessibility relation partitions the set of worlds into "clusters" or "cliques" of worlds that are internally fully connected but isolated from each other. Such equivalence classes are commonly discussed in tense logic, especially in connection with the notion of "historical necessity" of "settledness." We will see such an application in Kaufmann (2005).

6.6 System KD45

The last system we should consider consists of axioms we have already seen; to recapitulate:

(D) $\Box p \rightarrow \Diamond p$ (Consistency)

(4) $\Box p \rightarrow \Box\Box p$ (Positive Introspection)

(5) $\Diamond p \rightarrow \Box\Diamond p$ (Negative Introspection)

This system characterizes the class of frames that are *serial*, *transitive* and *euclidean*.

These are a bit like equivalence frames, but not quite. Intuitively: For each world w, the set of worlds accessible from w (i.e., the modal base for w) is an equivalence class, BUT w itself may not be a member of that equivalence class.

In a sense, therefore, KD45 is S5 minus the assumption of veridicality. This is particularly useful in modeling speakers' beliefs: They may be false, but in that case the speakers themselves are not aware that they are false. KD45 has been used in linguistics by Stalnaker, van Rooy and others in modeling speakers' beliefs and the common ground. We will see an example in Kaufmann (2005) (although there, the non-veridicality itself does not play an important role).

7 First-order predicate logic and modality

We will have little for this topic in these lectures. In particular, I expect that we won't be able to cover the formalities. If there is time, we'll go into Gamut it a bit: Modals and truth conditions are defined in Definitions 4 and 5 on pages 64 and 65 of the Gamut chapter.

7.1 Extension and intension

§1 The *extension* of an expression is its referent: Individuals for individual constants, sets of tuples for predicate constants; truth values for sentences.

§2 The *intension* of an expression is a function from worlds to extensions.

§3 The *Principle of Extensionality* mandates that two expressions with the same extension always be substituted in larger expressions without affecting the truth value.

§4 This Principle does not hold in *opaque* contexts, including modals. Below is an example.

7.2 De dicto and de re

Suppose the President dies in office. Then following inference is invalid.

(31) a. *[Under the constitution,]* the Vice President must take over.
b. Dick Cheney is the Vice President.
―――――――――――――――――――――――――――――
c. *[Under the constitution,]* Dick Cheney must take over.

§1 Definite descriptions like *'the Vice President'* have different extensions in different worlds: Al Gore might have been the Vice President, so there is an alternative world in which he *is* the VP.

§2 In contrast, the denotation of a proper name is *rigid*. It does not vary from world to world: There is no alternative world in which (our) Dick Cheney is (their) Al Gore. (There may be a world in which Dick Cheney looks, acts and thinks like Al Gore; but none in which he literally *is* Al Gore.)

The solution is to treat the difference as one of scope. There are two readings for (31a):

(32) a. For all worlds w' in the modal base, the individual x who is the Vice President at w' takes over in w'.
b. The individual x who is the Vice President at w is such that (s)he takes over in all worlds w' in the modal base.

a. The first is *de dicto*: It is about the *concept* of Vice President, not the individual who happens to play that role in our world.
b. The second *de re*: It is about the "thing" that is the VP in our world.

The Constitution stipulates (32a), not (32b). However, since proper names refer rigidly, the two scopes are semantically equivalent for (31c), and in our world, both are equivalent to (32b). That's why the inference in (32) is invalid.

Similar effects can be observed with temporal operators. The following are invalid:

(33) Ten years ago, the President was a Democrat.
George W. Bush is the President.
―――――――――――――――――――――
Ten years ago, George W. Bush was a Democrat.

(34) Ten years ago, the President was the Governor of Texas.
Ten years ago, Bill Clinton was the President.
―――――――――――――――――――――
Ten years ago, Bill Clinton was the Governor of Texas.

Scope interactions between quantifiers and modal operators are an important source of data about the structure of modalized sentence. We will see some of this later, in connection with the *epistemic/root* distinction. There is much more to be said about this topic, but we won't go into it here.

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